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LETTER TO THE EDITOR

Force on a vortex in ferromagnet model and the properties of vortex configurations

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Abstract. The general form is given for a force acting on a vortex in the ferromagnet model without any assumption for the profile of the vortex. The strength of the force is shown to be written in terms of the Mermin–Ho topological invariant for the vortex in superfluid He3-A. The discussions are given on the point of how the resulting formula can be interpreted as the mapping degree for the vortex configuration. These results are extended to the system described by the generalized spin.

1. Introduction

The purpose of this letter is to give a novel aspect for the result that has been obtained in our previous paper [1]. The problem concerns a specific force acting on a two-dimensional vortex in ferromagnets (or spin condensates), which we conventionally called the geometric force in our previous paper, since it is derived from the geometric part of the starting Lagrangian for the spin variable. In our previous paper, we calculated explicitly the force acting on the vortex in the ferromagnet model by adopting a very simple ansatz for the profile of the vortex, from which we showed that the force has the same property as the Magnus force. However, although that ansatz is reasonable for the derivation of the Magnus-type force, it is too simplified to study the general property of strength of the force.

In this letter, we consider the same force as that previously studied without recourse to any specific assumption concerning the profile function and give the general form for its strength. On the basis of this formula, we show an interesting similarity with the coreless vortex in superfluid He3-A, the famous Mermin–Ho relation [2, 3], and discuss how the strength can be interpreted as the mapping degree of the vortex configuration. The derivation and the discussion for a vortex in the ferromagnet model is easily applied to the $P_n(\mathbb{C})$ pseudospin model in the multi-layered quantum Hall ferromagnet. We give a brief sketch for this extension as an additional remark.

2. Preliminary

We start with a brief recapitulation of the formulation used in our previous paper [1]. As the starting Lagrangian for the ferromagnet spin system (two-dimensional case), we take

$$L = \int \left[\frac{J\hbar}{2}(1-\cos\theta)\dot{\phi} - H(\theta,\phi) \right] \mathrm{d}^2x \tag{1}$$

where the angle variables (θ, ϕ) are the spherical-coordinate parametrization of the spin field $J_{x,y,z}$:

$$J_x = J \sin\theta \cos\phi \qquad J_y = J \sin\theta \sin\phi \qquad J_z = J \cos\theta. \tag{2}$$

In equation (2), the first term is called the canonical term L_C, and the term H represents the Hamiltonian term in the continuous version of the anisotropic Heisenberg ferromagnet model; it is given by

$$H = \frac{g}{2} J^2 \int \{(\cos^2\theta + \lambda \sin^2\theta)(\nabla\theta)^2 + \sin^2\theta (\nabla\phi)^2\} \, \mathrm{d}^2 x \tag{3}$$

where the anisotropy parameter λ should be taken to satisfy $0 < \lambda \leqslant 1$, so as to favour the planar spin configuration [4–6]. A static solution for one vortex is obtained with the phase function $\phi = \tan^{-1}(y/x)$, and the profile function θ is given as a function of the radial variable r. The explicit form of the profile function $\theta(r)$ is derived from the extremum condition for the Hamiltonian H, with specific boundary conditions at $r = \infty$ and $r = 0$, which we will discuss later. Here we simply assume the existence of the solution $\theta(r)$.

3. Derivation of the force acting on a vortex

In order to treat the motion for a single vortex, we introduce the coordinate of the centre of the vortex, $\boldsymbol{X}(t) = (X(t), Y(t))$, by which the vortex solution is parametrized such that $\theta(\boldsymbol{x} - \boldsymbol{X}(t))$ and $\phi(\boldsymbol{x} - \boldsymbol{X}(t))$. We should make some comments about the effect of the collective degrees of freedom that are specific in quantum condensates such as phonon modes, which may cause dissipation of the vortex motion. As pointed out in [7], the phonon excitation around vortices has the effect of changing the profile, especially at finite temperatures. In the present letter, the low-energy limit is considered from the outset, so we neglect the coupling of phonon degrees of freedom. Although our treatment is rather restrictive, this modification plays no essential role in the topological feature of the force on vortices.

Now, using the parametrization prescribed in the above, the canonical term L_C, the first term in (1), is written as

$$L_C = \int \frac{J\hbar}{2} (1 - \cos\theta) \nabla\phi \cdot \dot{\boldsymbol{X}} \, \mathrm{d}^2 x \tag{4}$$

where we have used the relations

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial \boldsymbol{X}} \dot{\boldsymbol{X}} \qquad \frac{\partial\phi}{\partial \boldsymbol{X}} = -\nabla\phi. \tag{5}$$

Equation (4) shows that the momentum density conjugate to the coordinate $\boldsymbol{X}$ is given as

$$\boldsymbol{p} = \frac{J\hbar}{2} (1 - \cos\theta) \nabla\phi = \frac{J\hbar}{2} \boldsymbol{v} \tag{6}$$

where $\boldsymbol{v}$ is the velocity field

$$\boldsymbol{v} = (1 - \cos\theta) \nabla\phi. \tag{7}$$

In our previous paper [1], we have explicitly evaluated equation (4) with the simple ansatz for $\theta(r)$:

$$\theta(r) = \pi/2 (r > a) \qquad \text{and} \qquad \theta = \pi r / 2a (r < a)$$

and derived a force on a single vortex explicitly in that special case. In the present paper we do not use any specific form for the profile function $\theta(r)$. We will give a general form for the force on a single vortex, in particular, we show that it has a very similar structure to that of the vortex in superfluid He3. Keeping this in mind, we consider the Euler–Lagrange equation for $\boldsymbol{X}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\mathbf{X}}} - \frac{\partial L}{\partial \mathbf{X}} = 0. \tag{8}$$

We introduce the force $\boldsymbol{F}_C$ and $\boldsymbol{F}_H$ as follows:

$$\boldsymbol{F}_C = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_C}{\partial \dot{\boldsymbol{X}}} - \frac{\partial L_C}{\partial \boldsymbol{X}} \qquad \boldsymbol{F}_H = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial H}{\partial \dot{\boldsymbol{X}}} - \frac{\partial H}{\partial \boldsymbol{X}}. \tag{9}$$

By using $\boldsymbol{F}_{C,H}$, the Euler–Lagrange equation (8) can be written as the balance of forces: $\boldsymbol{F}_C = -\boldsymbol{F}_H$. $\boldsymbol{F}_C$ is the force that is called the geometric force in the previous paper, whereas $\boldsymbol{F}_H$ represents the force coming from the Hamiltonian H and is the same as the potential forces in the particle mechanics. In this paper, we are concerned with the structure of the specific force on a vortex, so we concentrate our attention on the force $\boldsymbol{F}_C$ in what follows. By using equation (6), we can obtain the general form for $\boldsymbol{F}_C$:

$$\boldsymbol{F}_C = \frac{J\hbar}{2}\left[\int\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathrm{d}^2x\right](\boldsymbol{k}\times\dot{\boldsymbol{X}}) \tag{10}$$

where $\boldsymbol{k}$ in (10) is the unit vector perpendicular to the xy-plane. In the derivation of (10), we have used the relation

$$\frac{\partial v_x}{\partial X} = -\frac{\partial v_x}{\partial x}. \tag{11}$$

Here we put the integral in (10) as

$$\sigma = \int\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathrm{d}^2x. \tag{12}$$

It should be noted that the integral σ in (12) does not depend on $\boldsymbol{X}$, because the integrand of (12) is a function of $\boldsymbol{x}-\boldsymbol{X}$, and with a change of variable $\boldsymbol{x}\to\boldsymbol{x}-\boldsymbol{X}$, σ becomes independent of $\boldsymbol{X}$.

4. Consideration using the vortex configurations

Now let us examine the meaning of σ. The integrand of σ is just the vorticity $\omega = \nabla\times\boldsymbol{v}$. Using equation (7), ω is expressed in terms of the spin field

$$\omega = \nabla\times\boldsymbol{v} = \sin\theta(\nabla\theta\times\nabla\phi) \tag{13}$$

and we obtain the final form for σ:

$$\sigma = \int_S \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi. \tag{14}$$

In equation (14), σ is the integral of a 2-form corresponding to the vorticity ω, which has a topological interpretation. Before discussing it, let us mention the relation between the vortex inherent in the ferromagnet model and a vortex in the A-phase of superfluid He3 (simply He3-A). The ω in (13) can be rewritten with the spin field $\boldsymbol{S}$ normalized to one ($\boldsymbol{S} = \boldsymbol{J}/J$):

$$\nabla\times\boldsymbol{v} = \boldsymbol{S}\cdot\left(\frac{\partial\boldsymbol{S}}{\partial x}\times\frac{\partial\boldsymbol{S}}{\partial y}\right) \tag{15}$$

which can be checked easily by using equation (2). Equation (15) is just the Mermin–Ho relation for the spin condensate, which was originally obtained for He3-A [2]. Note that

$\boldsymbol{S}$ corresponds to the l vector in He3-A. The reason for the appearance of the Mermin–Ho relation for the spin vortex originates in the similar vector structure for both variables in He3-A and the spin ferromagnet.

To discuss the meaning of σ, we consider the boundary conditions for the profile function $\theta(r)$. At the origin $r = 0$, we can take $\theta(0) = 0$ in general. On the other hand, at $r = \infty$, there exist several possibilities, but we consider two typical cases here: (A) $\theta(\infty) = \pi$ and (B) $\theta(\infty) = \pi/2$. (In the A-phase of superfluid He3, (A) and (B) correspond to the Anderson–Toulouse and the Mermin–Ho vortices, respectively [2, 3].) In both cases, $S_z(0) = 1$ and the spin field $\boldsymbol{S}(x)$ are directed upwards, whereas, at $r = \infty$, $S_z(\infty) = -1$ for case (A) and $S_z(\infty) = 0$ for case (B): the spin field is directed *downwards* for (A) and *outwards* for (B). In case (A), the vortex configuration can be considered as the continuous mapping from the compactified space S_2 to the spin configuration S_2, and the σ in (14) has a clear topological meaning: the mapping degree of $S_2 \longrightarrow S_2$. As a result, we get the quantization of σ: $\sigma = m$ (m = integer). In case (B), we cannot consider the space as a compactified S_2 because of the boundary condition at $r = \infty$, but it is rather a hemisphere D_2 with the boundary circle. Now what we have to consider is the mapping from D_2 to S_2, and no topological invariants exist generally. However, all vortex configurations in case (B) have a common special boundary condition from the boundary circle to the fixed S_1 specified by $S_z = 0$ in the spin configuration. This situation allows the interpretation of the mapping degree for σ in terms of the concept of the 'relative homotopy group' $\pi_2(S_2, S_1)$ in the terminology of topology: the homotopy class of mappings from the disc D_2 to the sphere S_2, where the boundary circle S_1 is mapped on the subspace S_1. (The standard homotopy group $\pi_2(S_2)$ is one of the mappings where the boundary S_1 is mapped onto any point on S_2.) We still get the quantization of σ based on it, namely, $\sigma = m/2$ (m = integer). Concerning the more details of the relative homotopy group, see [8].

Summarizing the above discussions, the force $\boldsymbol{F}_C$ is written in terms of the mapping degree of freedom:

$$\boldsymbol{F}_C = \frac{J\hbar}{2}\sigma(\boldsymbol{k} \times \dot{\boldsymbol{X}}). \tag{16}$$

As discussed in [1], the force $\boldsymbol{F}_C$ corresponds to a Lorentz force acting on a charged particle in magnetic field, which is of non-dissipative nature. Alternatively, it may be regarded as an analogue of the Magnus force in superfluids or superconductors [9, 10]. However, this interpretation is not so exact physically, because the Magnus force is defined to be that acting on a vortex in the uniform superflow and we do not have superflow in the ferromagnet model which we have considered in this paper.

5. Additional remarks

We briefly sketch a possible generalization of equation (16) to a model defined by the complex projective space $P_n(\mathbb{C}) = U(n+1)/U(n) \times U(1)$, which can be realized for the generalized pseudospin in the multi-layered quantum Hall ferromagnet [11]. The spin ferromagnet model is included as a special case of the $P_n(\mathbb{C})$, because it is the model on the sphere $S_2 \sim P_1(\mathbb{C})$. The $P_n(\mathbb{C})$ model is described by the n-component complex vector field, $\xi(x) = (z_1(x), \ldots, z_n(x))$, and the canonical term is given by

$$L_C = \frac{\mathrm{i}\hbar}{2}\int \sum_{i=1}^{n}\left(\frac{\partial \log F}{\partial z_i}\dot{z}_i - \mathrm{cc}\right)\mathrm{d}^2x \tag{17}$$

where $F = (1 + \xi^\dagger(x)\xi(x))^N$ is called the kernel function (N = integer). If we use the collective-coordinate parametrization $\xi(\boldsymbol{x} - \boldsymbol{X}(t))$, equation (17) becomes

$$L_C = \frac{\hbar}{2} \int \boldsymbol{p} \cdot \dot{\boldsymbol{X}} \, \mathrm{d}^2 x \tag{18}$$

where the momentum $\boldsymbol{p}$ is given by

$$\boldsymbol{p} = \mathrm{i} \sum_{i=1}^{n} \left\{ \frac{\partial \log F}{\partial z_i} \nabla z_i - \mathrm{CC} \right\}. \tag{19}$$

The force $\boldsymbol{F}_C$ acting on a vortex can be calculated from (18) following the same procedure as in (10):

$$\boldsymbol{F}_C = \frac{\hbar}{2} \sigma (\boldsymbol{k} \times \dot{\boldsymbol{X}}) \tag{20}$$

where σ becomes

$$\sigma = \int_{R^2} \sum_{ij} g_{i\bar{j}} \left\{ \frac{\partial z_i}{\partial x} \frac{\partial z_j^*}{\partial y} - (x \to y) \right\} \mathrm{d}x \, \mathrm{d}y = \int_{\tilde{X}} \sum_{ij} g_{i\bar{j}} \, \mathrm{d}z_i \wedge \mathrm{d}z_j^*. \tag{21}$$

$g_{i\bar{j}}$ in (21) is the Kähler metric that is given by

$$g_{i\bar{j}} = \frac{\partial^2 \log F}{\partial z_i \partial z_j^*}. \tag{22}$$

It is also possible to give a similar interpretation for this integral σ as in the case of the ferromagnet model.

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